The Dirichlet boundary problems for second order parabolic operators satisfying a Carleson condition

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Definition of Solvability

For $1 < p \leq \infty$ and $\Omega$ (an admissible Lipschitz domain), consider the parabolic Dirichlet boundary value problems

\[
\begin{aligned}
0 &= \sum_{i,j} \partial_{x_i} (a_{ij}(x) \partial_{x_j} u(x)) \quad \text{in } \Omega, \\
u &= f \in L^p(\partial\Omega) \quad \text{on } \partial\Omega, \\
N(u) &\in L^p(\partial\Omega, d\sigma).
\end{aligned}
\] (1)

where $A$ satisfies the uniform ellipticity conditions. We say (1) is solvable if there exists a constant $C$ such that

\[
\|N(u)\|_{L^p(\partial\Omega, d\sigma)} \leq C\|f\|_{L^p(\partial\Omega, d\sigma)}
\]

where $C$ depends on the operator, $p$, and the domain.
Negative Results

Theorem

Consider

\[ 0 = \sum_{i,j} \partial_{x_i} (a_{ij}(x) \partial_{x_j} u(x)) . \]

There exists a bounded measurable coefficient matrix \( A = [a_{ij}] \) on a unit disk \( D \) satisfying the uniform ellipticity conditions such that the \( L^p \)–Dirichlet problem is not solvable for any \( p \in (1, \infty) \).
Negative Results

Theorem

Consider

\[ 0 = \sum_{i,j} \partial_{x_i} \left( a_{ij}(x) \partial_{x_j} u(x) \right). \]

There exists a bounded measurable coefficient matrix \( A = [a_{ij}] \) on a unit disk \( D \) satisfying the uniform ellipticity conditions such that the \( L^p \)-Dirichlet problem is not solvable for any \( p \in (1, \infty) \).

- It is difficult because both the coefficients and the domain are given rough.
- We impose minimal and natural smoothness conditions (so called Carleson conditions) on the coefficients.
$L^p$ Dirichlet problems

Definition of Solvability

For $1 < p \leq \infty$ and $\Omega$ (an admissible Lipschitz domain), consider the parabolic Dirichlet boundary value problems

\[
\begin{cases}
  u_t = \text{div} (A \nabla u) + B \cdot \nabla u & \text{in } \Omega, \\
  u = f \in L^p(\partial\Omega) & \text{on } \partial\Omega, \\
  N(u) \in L^p(\partial\Omega, d\sigma). 
\end{cases}
\]

where $A$ satisfies the uniform ellipticity conditions. We say (2) is solvable if there exists a constant $C$ such that

\[
\|N(u)\|_{L^p(\partial\Omega, d\sigma)} \leq C\|f\|_{L^p(\partial\Omega, d\sigma)}
\]

where $C$ depends on the operator, $p$, and the domain.
Parabolic distance and ball

- Parabolic distance:
  \[
  \delta(X, t) = \inf_{(Y, s) \in \partial \Omega} \left( |X - Y| + |t - s|^{1/2} \right).
  \]

- Parabolic ball:
  \[
  B_r(X, t) = \{(Y, s) \in \mathbb{R}^n \times \mathbb{R} : |X - Y| + |t - s|^{1/2} < r\}.
  \]

- Surface ball and Corresponding Interior:
  \[
  \Delta_r(X, t) = \partial \Omega \cap B_r(X, t),
  \]
  \[
  T(\Delta_r)(X, t) = \Omega \cap B_r(X, t).
  \]
Carleson Measure

Definition

Then a measure $\mu : \Omega \to \mathbb{R}^+$ is said to be Carleson if there exists a constant $C = C(r_0)$ such that for all $r \leq r_0$ and all surface balls $\Delta_r$

$$\mu(T(\Delta_r)) \leq C\sigma(\Delta_r).$$

The best constant $C(r_0)$ is called the Carleson norm. If $\lim_{r_0 \to 0} C(r_0) = 0$, then we say that the measure $\mu$ satisfies the vanishing Carleson condition.
Assume that

$$\delta(X, t)^{-1} \left( \text{osc}_{B\delta(X,t)/2}(x,t) a_{ij} \right)^2 + \delta(X, t) \left( \sup_{B\delta(X,t)/2}(x,t) b_i \right)^2$$

is the density of Carleson measure with small Carleson norm.
Assume that
\[
\delta(X, t)^{-1} \left( \text{osc}_{B\delta(X, t)/2}(X, t) a_{ij} \right)^2 + \delta(X, t) \left( \sup_{B\delta(X, t)/2}(X, t) b_i \right)^2
\]
is the density of Carleson measure with small Carleson norm.

Small Carleson conditions on the coefficients

\[
(|\nabla A|^2 \delta(X, t) + |A_t|^2 \delta^3(X, t)) \ dX \ dt, \quad |B|^2 \delta(X, t) \ dX \ dt
\]
are Carleson measures with the norm \( \epsilon \).
For $X = (x_0, x) \in \mathbb{R} \times \mathbb{R}^{n-1}$, consider the region given by

$$\Omega = \{(x_0, x) : x_0 > \psi(x)\}$$

where the function $\psi$ satisfies the followings

$$|\psi(x) - \psi(y)| \leq L|x - y|.$$
The time-varying Lipschitz domain

For $X = (x_0, x) \in \mathbb{R} \times \mathbb{R}^{n-1}$, consider the region given by

$$\Omega = \{(x_0, x, t) : x_0 > \psi(x, t)\}$$

where the function $\psi$ satisfies the followings

$$|\psi(x, t) - \psi(y, s)| \leq L \left(|x - y| + |t - s|^{1/2}\right).$$

- **Hunt’s conjecture:** Lip($1, 1/2$) is proper domain for parabolic equations
The time-varying Lipschitz domain

For \( X = (x_0, x) \in \mathbb{R} \times \mathbb{R}^{n-1} \), consider the region given by

\[
\Omega = \{(x_0, x, t) : x_0 > \psi(x, t)\}
\]

where the function \( \psi \) satisfies the followings

\[
|\psi(x, t) - \psi(y, s)| \leq L \left( |x - y| + |t - s|^{1/2} \right).
\]

- Hunt’s conjecture: Lip\((1, 1/2)\) is proper domain for parabolic equations
- Kaufmann & Wu (1988) : With only Lip\((1, 1/2)\), parabolic measure and surface measure are failed to be mutually absolutely continuous
The admissible time-varying Lipschitz domain

For $X = (x_0, x) \in \mathbb{R} \times \mathbb{R}^{n-1}$, consider the region given by

$$\Omega = \{(x_0, x, t) : x_0 > \psi(x, t)\}$$

where the function $\psi$ satisfies the followings

$$|\psi(x, t) - \psi(y, s)| \leq L \left(|x - y| + |t - s|^{1/2}\right),$$

$$\|D_{1/2}^t \psi\|_{\text{BMO}} \leq L.$$

- Lewis & Murray (1995) : Establish mutual absolute continuity of caloric measure and a certain parabolic analogue of surface measure

- An admissible parabolic is defined locally satisfying Lip(1, 1/2) and ‘Lewis-Murray’ conditions using at most $N L$—cylinders.
Non-tangential maximal functions

- For \((x_0, x, t) \in \partial \Omega\), define a parabolic cone

\[
\Gamma_a(x_0, x, t) = \{(y_0, y, s) \in \Omega : |x - y| + |s - t|^{1/2} \leq a(y_0 - x_0), y_0 > x_0\}.
\]

- Non-tangential maximal functions:

\[
N_a(u)(x_0, x, t) = \sup_{(y_0, y, s) \in \Gamma_a(x_0, x, t)} |u(y_0, y, s)|.
\]
Non-tangential maximal function and Carleson measure

We have integral inequalities that

\[ \int_{T(\Delta_r)} \left( \delta |\nabla A|^2 + \delta^3 |A_t|^2 + \delta |B|^2 \right) u^2 \, dX \, dt \leq \| \mu \|_C \int_{\Delta_r} N^2(u) \, dx \, dt \]

where \( \| \mu \|_C \) is a Carleson norm.
- Dahlberg (1977): Elliptic problem with $L^2$ Dirichlet boundary data
- Kenig, Koch, Pipher, Toro (2000): Elliptic operator satisfying Carleson measure condition, Comparison of the square and non-tangential maximal functions
- Dindos, Petermichl, Pipher (2007): $L^p$ Dirichlet Elliptic problems under small Carleson conditions, Quantitative approach

- Symmetry of the coefficients is NOT assumed.
- Does NOT rely on layer potentials rather direct approach.
- Compare the non-tangential maximal function and the square function
Pullback Transformation

\[ v_t = \text{div}(\tilde{A} \nabla v) + \tilde{B} \cdot \nabla v \]

\[ \Omega = \{(x_0, x, t) : x_0 > \psi(x, t)\} \]

\[ u_t = \text{div}(A \nabla u) + B \cdot \nabla u \]

\[ U = \{(x_0, x, t) : x_0 > 0\} \]

pullback transformation
\[ u = v \circ \rho \]
\[ \rho(x_0, x, t) = (x_0 + P_{x_0} \psi(x, t), x, t) \]
Pullback Transformation

- \( x_0 \) independent transformation
  It is trivial to use
  \[
  \rho(x_0, x, t) = (x_0 + \psi(x, t), x, t).
  \]
  But \( v_t \) provides \( \psi_t(x, t)u_{x_0}(x_0, x, t) \) where \( \psi_t \) may not defined because of the lack of regularity.

- For Elliptic equations, corresponding coefficient \( A \) is \( x_0 \) independent. (Kenig, Koch, Pipher, & Toro 2000, 2 dimension for Dirichlet problem, Hofmann, Kenig, Mayboroda, & Pipher Higher dimensional Dirichlet problem, Rellich inequality)
Dalhberg-Kenig-Necas-Stein transformation using mollification

We define

$$\rho(x_0, x, t) = (x_0 + P_{\gamma x_0} \psi(x, t), x, t).$$

For a nonnegative function $P(x, t) \in C_0^\infty$ for $(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, set

$$P_\lambda(x, t) \equiv \lambda^{-(n+1)} P(\lambda^{-1} x, \lambda^{-2} t),$$

$$P_\lambda \psi(x, t) \equiv \int_{\mathbb{R}^{n-1} \times \mathbb{R}} P_\lambda(x - y, t - s) \psi(y, s) \, dy \, ds.$$

**Elliptic Cases:** basically allows Layer potentials (Mitrea & Taylor 2003, Hofamman & Lewis 2001 Double layer potentials for parabolic equations)

**We avoid layer potential methods.**
Definitions

Square functions:

\[
S_a(u)(x_0, x, t) = \left( \int_{\Gamma_a(x_0, x, t)} |\nabla u|^2 \, \delta^{-n}(y_0, y, s) \, dy_0 \, dy \, ds \right)^{1/2},
\]

\[
\|S_a(u)\|_{L^2(\partial\Omega)}^2 = \int_{\Omega} |\nabla u|^2 \, \delta(y_0, y, s) \, dy_0 \, dy \, ds
\]
Definitions

- **Square functions:**

\[
S_a(u)(x_0, x, t) = \left( \int_{\Gamma_a(x_0, x, t)} |\nabla u|^2 \delta^{-n}(y_0, y, s) \, dy_0 \, dy \, ds \right)^{1/2},
\]

\[
\|S_a(u)\|^2_{L^2(\partial \Omega)} = \int_\Omega |\nabla u|^2 \delta(y_0, y, s) \, dy_0 \, dy \, ds,
\]

- **Area functions:**

\[
A_a(u)(x_0, x, t) = \left( \int_{\Gamma_a(x_0, x, t)} |u_t|^2 \delta^{-n+2}(y_0, y, s) \, dy_0 \, dy \, ds \right)^{1/2},
\]

\[
\|A_a(u)\|^2_{L^2(\partial \Omega)} = \int_\Omega |u_t|^2 \delta^3(y_0, y, s) \, dy_0 \, dy \, ds.
\]
Strategy

- $L^\infty$: the Maximum Principle
- $L^p$ for $2 < p < \infty$: Interpolation if we have $L^2$-solvability
Strategy

- $L^\infty$: the Maximum Principle
- $L^p$ for $2 < p < \infty$: Interpolation if we have $L^2$-solvability
- $L^2$: Quantitative method comparing $\|S(u)\|_{L^2(\partial\Omega)}$ and $\|N(u)\|_{L^2(\partial\Omega)}$
**Definition of Solvability**

For $\Omega$ (an admissible Lipschitz domain), consider the parabolic Dirichlet boundary value problems

\[
\begin{aligned}
&u_t = \text{div} (A \nabla u) + B \cdot \nabla u \quad \text{in } \Omega, \\
&u = f \in L^2(\partial \Omega) \quad \text{on } \partial \Omega, \\
&N(u) \in L^2(\partial \Omega, d\sigma).
\end{aligned}
\]

where $A$ satisfies the uniform ellipticity conditions. We say (3) is solvable if there exists a constant $C$ such that

\[
\|N(u)\|_{L^2(\partial \Omega, d\sigma)} \leq C \|f\|_{L^2(\partial \Omega, d\sigma)}
\]

where $C$ depends on the operator and the domain.
**$L^2$ Dirichlet problem**

- **Uniform Ellipticity and Boundedness** for $A = [a_{ij}]$
  \[ \lambda |\xi|^2 \leq \sum a_{ij} \xi_i \xi_j \leq \Lambda |\xi|^2. \]

- **Small Carleson conditions** on the coefficients
  \[
  (|\nabla A|^2 \delta(X, t) + |A_t|^2 \delta^3(X, t)) \, dX \, dt,
  |B|^2 \delta(X, t) \, dX \, dt
  \]
  are Carleson measures with the norm $\epsilon$.

- **Boundedness from the Carleson conditions**
  \[
  |\nabla A| \delta(X, t) + |A_t| \delta^2(x, t) \leq \epsilon^{1/2},
  |B| \delta(X, t) \leq \epsilon^{1/2}.
  \]
Rough Sketch

It is the results of three estimates.

- Comparing $A(u)$ and $S(u)$ using Caccioppoli’s inequality
It is the results of three estimates.

- Comparing $A(u)$ and $S(u)$ using Caccioppoli’s inequality
- First,

\[
\int_{\partial \Omega} S^2(u) \, dx \, dt \leq C \int_{\partial \Omega} f^2 \, dx \, dt + \epsilon \int_{\partial \Omega} N^2(u) \, dx \, dt
\]

where $\epsilon$ comes from the smallness of the Carleson measure of the coefficients.
Rough Sketch

It is the results of three estimates.

- Comparing $A(u)$ and $S(u)$ using Caccioppoli’s inequality

  First,

  $$\int_{\partial \Omega} S^2(u) \, dx \, dt \leq C \int_{\partial \Omega} f^2 \, dx \, dt + \epsilon \int_{\partial \Omega} N^2(u) \, dx \, dt$$

  where $\epsilon$ comes from the smallness of the Carleson measure of the coefficients.

- The second inequality is given by

  $$\int_{\partial \Omega} N^2(u) \, dx \, dt \leq C \left[ \int_{\partial \Omega} S^2(u) \, dx \, dt + \int_{\partial \Omega} f^2 \, dx \, dt \right].$$
We replace $\|S(u)\|_{L^2(\partial \Omega)}$ by the first inequality. Then

$$
\int_{\partial \Omega} N^2(u) \, dX \, dt \leq C \left[ \int_{\partial \Omega} S^2(u) \, dX \, dt + \int_{\partial \Omega} f^2 \, dX \, dt \right]
$$

$$
\leq C' \int_{\partial \Omega} f^2 \, dX \, dt + \epsilon \int_{\partial \Omega} N^2(u) \, dX \, dt.
$$
We replace $\|S(u)\|_{L^2(\partial \Omega)}$ by the first inequality. Then

$$
\int_{\partial \Omega} N^2(u) \, dX \, dt \leq C \left[ \int_{\partial \Omega} S^2(u) \, dX \, dt + \int_{\partial \Omega} f^2 \, dX \, dt \right] \\
\leq C' \int_{\partial \Omega} f^2 \, dX \, dt + \epsilon \int_{\partial \Omega} N^2(u) \, dX \, dt.
$$

Because of $\epsilon$, we can hide $\|N(u)\|_{L^2(\partial \Omega)}$. Therefore, it follows

$$
\|N(u)\|_{L^2(\partial \Omega)} \leq C \|f\|_{L^2(\partial \Omega)}.
$$
Theorem

For an admissible parabolic domain \( \Omega \), let \( A = [a_{ij}] \) satisfy the uniform ellipticity and boundedness with constants \( \lambda \) and \( \Lambda \). In addition, assume that

\[
d\mu = \left[ \delta(X, t)^{-1} \sup_{1 \leq i, j \leq n} \left( \frac{\text{osc}_{B_{\delta(X, t)/2}(X, t)} a_{ij}}{\delta(X, t)} \right)^2 + \delta(X, t) \sup_{B_{\delta(X, t)/2}(X, t)} |B|^2 \right] dX \, dt
\]

is the density of a Carleson measure on \( \Omega \) with Carleson norm \( \| \mu \|_C \).

Continue..
Theorem

Then there exists $\varepsilon > 0$ such that if for some $r_0 > 0$ $\max\{L, \|\mu\|_{C,r_0}\} < \varepsilon$ then the $L^p$ boundary value problem

$$
\begin{cases}
  u_t = \text{div}(A\nabla u) + B \cdot \nabla u & \text{in } \Omega, \\
  u = f \in L^p & \text{on } \partial\Omega, \\
  N(u) \in L^p(\partial\Omega),
\end{cases}
$$

is solvable for all $1 \leq p < \infty$. Moreover, the estimate

$$
\|N(u)\|_{L^p(\partial\Omega,d\sigma)} \leq C_p \|f\|_{L^p(\partial\Omega,d\sigma)},
$$

holds with $C_p = C_p(L, \|\mu\|_{C,r_0}, N, \lambda, \Lambda)$. 

Hwang Carleson condition
Solvability for Elliptic system

- Joint work with M. Dindos and M. Mitrea (U of Missouri)
- Provide quantitative method for Elliptic system with $L^2$—Dirichlet boundary condition.
- The lack of regularity such as no Maximum principle. Not always possible to obtain $L^p$ solvability for $p > 2$.
- With Regularity problem and Reverse Hölder inequality, in fact, we have results on $L^p$—Dirichlet problem for $2 \leq p < p^*$ where $p^*$ depending on $n$ and $p$. 

Hwang Carleson condition
Definition

Let $\Omega$ be the Lipschitz domain $\{(x_0, x') : x_0 > \phi(x')\}$. Let $1 < p < \infty$. Consider the following Dirichlet problem for a vector valued function $u : \Omega \to \mathbb{R}^n$

$$
\begin{cases}
0 = \left[ \partial_i \left( A_{ij}^{\alpha\beta}(x) \partial_j u_{\beta} \right) + B_i^{\alpha\beta}(x) \partial_i u_{\beta} \right]_{\alpha} & \text{in } \Omega, \quad \alpha = 1, 2, \ldots, N \\
u(x) = f(x) & \text{on } \partial\Omega, \\
\tilde{N}(u) \in L^p(\partial\Omega).
\end{cases}
$$

(6)

Here we use the usual Einstein convention and sum over repeating indices ($i$ and $\beta$). We say the Dirichlet problem (6) is solvable for a given $p \in (1, \infty)$, if there exists $C = C(\lambda, \Lambda, n, p, \Omega) > 0$ such that the unique energy solution ($u \in W^{1,2}(\Omega)$ obtained via Lax-Milgram lemma) with boundary data $f \in L^p(\partial\Omega) \cap B_{1/2}^{2,2}(\partial\Omega)$ satisfies the estimate

$$
\|\tilde{N}u\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}.
$$

(7)
Definition

For $\Omega \subset \mathbb{R}^n$ as above the nontangential maximal function of $u$ at $Q \in \partial \Omega$ relative to the cone $\Gamma_a(Q)$ and its truncated version at height $h$ are defined by

$$N_a(u)(Q) = \sup_{x \in \Gamma_a(Q)} |u(x)| \quad \text{and} \quad N_a^h(u)(Q) = \sup_{x \in \Gamma_a^h(Q)} |u(x)|.$$ 

Moreover, we shall also consider a weaker version $\tilde{N}$ of the nontangential maximal function that is defined using $L^2$ averages over balls in the domain $\Omega$. We set

$$\tilde{N}_a(u)(Q) = \sup_{x \in \Gamma_a(Q)} w(x) \quad \text{and} \quad \tilde{N}_a^h(u)(Q) = \sup_{x \in \Gamma_a^h(Q)} w(x),$$

where

$$w(x) := \left( \frac{1}{|B_{\delta(x)/2}(x)|} \int_{B_{\delta(x)/2}(x)} u^2(z) \, dz \right)^{1/2}. \quad (8)$$
Theorem (assumptions)

Let $\Omega$ be the Lipschitz domain $\{(x_0, x') : x_0 > \phi(x')\}$ with Lipschitz constant $L = \|\nabla \phi\|_{L^\infty}$. Assume that the coefficients $A$ of the system (6) are strongly elliptic with parameters $\lambda, \Lambda$. In addition assume that at least one of the following holds.

(i) $\Omega = \mathbb{R}^n_+$ and the matrix $A_{00} = \left( A_{\alpha\beta}^{\alpha\beta} \right)_{\alpha,\beta} = I_{N \times N}$ on $\Omega$.

(ii) For all $i, j, \alpha, \beta$ we have $A_{ij}^{\alpha\beta} = A_{ij}^{\beta\alpha}$ on $\Omega$. 
Then there exists $K = K(\lambda, \Lambda, n) > 0$ such that if

$$d\mu(x) = \left[ \left( \sup_{B_{\delta(x)/2}(x)} |\nabla A(x)| \right)^2 + \left( \sup_{B_{\delta(x)/2}(x)} |B(x)| \right)^2 \right] \delta(x) \, dx \quad (9)$$

is a Carleson measure with norm $\|\mu\|_C$ less than $K$ and the Lipschitz constant $L < K$ the the $L^2$ Dirichlet problem for the system $(6)$ is solvable and the estimate

$$\|\tilde{N}u\|_{L^2(\partial\Omega)} \leq C\|f\|_{L^2(\partial\Omega)}.$$

holds for all energy solutions $u$ with $f \in L^2(\partial\Omega) \cap B^{2,2}_{1/2}(\partial\Omega)$ where $C(\lambda, \Lambda, n, \Omega) > 0$. 
Thank you for your attention!!