Degree counting for Toda system of rank two: one bubbling

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The 1st Meeting of Young Researchers in PDEs
Yonsei University, Seoul, Korea
National Institute for Mathematical Sciences.

Aug. 18-19, 2016
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Mean field equation

\( (M, g) \) : a compact Riemann surface with volume 1.

\( \Delta \) : Laplace-Beltrami operator.

\( \bullet \) Mean field equation:

\[
\Delta u^* + \rho \left( \frac{h^* e^{u^*}}{\int_M h^* e^{u^*}} - 1 \right) = 4\pi \sum_{q \in S_0} \alpha_q (\delta_q - 1) \text{ in } M \quad (\text{MFE}).
\]

\( h^* \in C^{2,\sigma}(M), \quad h^* > 0, \quad \alpha_q > -1, \quad \rho > 0. \)

\( \delta_q \): Dirac measure at \( q \).

\( q \in S_0 \) : vortex point (or singular source).
Background

\[ \Delta u_* + \rho \left( \frac{h_* e^{u_*}}{\int_M h_* e^{u_*}} - 1 \right) = 4\pi \sum_{q \in S_0} \alpha_q (\delta_q - 1) \text{ in } M \quad (MFE). \]

- Geometry:
  
  \( S_0 = \emptyset \): the Nirenberg problem of prescribed Gaussian curvature,

  \( S_0 \neq \emptyset \): the existence of a positive constant curvature metric with conic singularities.

- Physics: abelian gauge field theory.

- Statistical physics: (MFE) is obtained from the mean field limit of point vortices of Euler flows or spherical Onsager vortex theory.

- Biology: Keller-Segel model

- Lame equation and Painleve equation.
Background (Conformal geometry)

• $(M, g)$: a compact Riemann surface with volume 1.

$K_g$: Gaussian curvature of $(M, g)$

• $\bar{g}$ is pointwise conformal to $g$ if $\bar{g} = \rho g$ for some $\rho > 0$.

A conformal map preserves angles but can change lengths.

Denote $\rho = e^{2w}$ and $g_w = e^{2w} g$.

• The behavior at a conic singularity $p_j \in S_0$ is $e^w(z) \sim |z|^{\theta_j - 1}$, where $z$ is the local coordinate which is 0 at $p_j$, and $2\pi \theta_j > 0$ is the total angle around the singularity $p_j$.

$$\Delta_{g_w} + K_{g_w} e^{2w} - K_g = \sum_{p_j \in S_0} 2\pi (\theta_j - 1) \delta_{p_j} \text{ in } M.$$
Background (Chern-Simons field theory)

- Chern-Simons field theories have been developed to study high temperature superconductivity. Abelian Chern-Simons models was proposed by [J. Hong, Y. Kim, P.Y. Pac (1990)] and [R. Jackiw, E.J. Weinberg (1990)].

- Chern-Simons Higgs model can be reduced to

\[
\Delta u_\varepsilon + \frac{1}{\varepsilon^2} e^{u_\varepsilon} (1 - e^{u_\varepsilon}) = 4\pi \sum_{j=1}^{N} \delta_{p_j} \quad (CSH).
\]

Theorem (K. Choe, N. Kim (2008))

Let \( u_\varepsilon \) be solutions of (CSH) on a torus \( \mathbb{T} \). Then

(i) \( \lim_{\varepsilon \to 0} \left( \sup_K |u_\varepsilon| \right) = 0, \forall K \subseteq \mathbb{T} \setminus \{p_j\} \); or

(ii) \( u_\varepsilon - 2 \ln \varepsilon \) is uniformly bounded in \( L^\infty_{loc}(\mathbb{T} \setminus \{p_j\}) \); or

(iii) \( \exists \) a finite blow up points set \( S \neq \emptyset \subseteq \mathbb{T} \).

If (ii) occurs, then \( u_\varepsilon - 2 \ln \varepsilon \to \omega \), where \( \omega \) is a solution of (MFE).
Reduced equation without singular sources

• $G(x, y)$: Green function on $M$.

\[ \Delta G(x, p) = -\delta_p + 1 \text{ in } M, \quad \text{and } \int_M G(x, p) = 0. \]

• Denote $u_*(x) = u(x) - 4\pi \sum_{q \in S_0} \alpha_q G(x, q)$.

\[ \Delta u_* + \rho \left( \frac{h_* e^{u_*}}{\int_M h_* e^{u_*}} - 1 \right) = 4\pi \sum_{q \in S_0} \alpha_q (\delta_q - 1) \text{ in } M \quad (MFE). \]

\[ \iff \]

\[ \Delta u + \rho \left( \frac{he^u}{\int_M he^u} - 1 \right) = 0 \text{ in } M \quad (MFE)_0, \]

where $h(x) = h_*(x)e^{-\sum_{q \in S_0} 4\pi \alpha_q G(x, q)} \geq 0$ in $M$.

\[ h(x) \sim |x - q|^{2\alpha_q} \text{ for } |x - q| \ll 1. \]
Normalization

\[ \Delta u + \rho \left( \frac{he^u}{\int_M he^u} - 1 \right) = 0 \text{ in } M \quad (MFE)_0, \]

where \( h(x) = h_*(x)e^{-\sum_{q \in S_0} 4\pi \alpha_q G(x,q)} \).

- \((MFE)_0\) is invariant by adding a constant to the solutions.

- We always consider \((MFE)_0\) in the following function space:

\[ \hat{H}^1(M) = \left\{ u \in H^1(M) \mid \int_M u = 0 \right\}. \]
A priori bounds of solutions for $(MFE)_0$ in $\dot{H}^1(M)$

\[
\Delta u + \rho\left(\frac{he^u}{\int_M he^u} - 1\right) = 0 \text{ in } M \quad (MFE)_0,
\]

where \( h(x) = h_*(x)e^{-\sum_{q \in S_0} 4\pi \alpha_q G(x,q)} \).

- The set of critical parameters

\[
\Sigma : = \{8N\pi + \Sigma_{p \in A} 8\pi (1 + \alpha_p) \mid N \in \mathbb{N} \cup \{0\}, \ A \subseteq S_0\} \setminus \{0\}.
\]

**Theorem** (Brezis-Merle (1991), Li-Shafrir (1994), Bartolucci-Tarantello (2002))

Let \( \rho \notin \Sigma \). Then all solutions of $(MFE)_0$ in $\dot{H}^1(M)$ are uniformly bounded.
Leray-Schauder degree $d_{\rho}$

- Compact operator

$$T_{\rho}u = \rho \Delta^{-1} \left( \frac{he^{u}}{\int_{M} he^{u} dv_{g}} - 1 \right).$$

- By [Brezis-Merle (1991), Li-Shafrir (1994), Bartolucci-Tarantello (2002)], the Leray-Schauder degree

$$d_{\rho} := \deg(I + T_{\rho}, B_{R}, 0)$$

is well defined for $\rho \notin \Sigma$, where $B_{R} = \{ u \in \dot{H}^{1}(M) \mid \| u \|_{H^{1}(M)} \leq R \}$.

- $\Sigma = \{ 8\pi a_{j} \mid a_{1} \leq a_{2} \cdots \}$. 

Since the Leray-Schauder degree $d_{\rho}$ is a homotopic invariant for $\rho \in (8a_{j}\pi, 8a_{j+1}\pi)$,

$$d_{\rho} \equiv d_{j}.$$. 

Leray-Schauder degree $d_\rho$

- The set of critical parameters

$$\Sigma = \{8N\pi + \sum_{p \in A} 8\pi(1 + \alpha_p) \mid N \in \mathbb{N} \cup \{0\}, \ A \subseteq S_0 \} \setminus \{0\}$$

$$= \{8\pi a_j \mid a_1 \leq a_2 \cdots \}.$$

From $d_\rho \equiv d_j$ for $\rho \in (8a_j\pi, 8a_{j+1}\pi)$, we denote the generating function by

$$g^{(1)}(x) := \sum_{j=0}^{\infty} d_j x^j.$$


$$g^{(1)}(x) = (1 - x)^{\chi(M) - |S_0| - 1} \prod_{p \in S_0} \left(1 - x^{1+\alpha_p}\right),$$

where $\chi(M)$ is the Euler characteristic of $M$. 
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Toda system

- Consider solution \( u = (u_1, \cdots, u_n) \) of the following system

\[
\Delta u_i + \sum_{j=1}^{n} K_{ij} \rho_j \left( \frac{h_j^* e^{u_j}}{\int_M h_j e^{u_j} dv} - 1 \right) = \sum_{s=1}^{N} 4\pi \alpha_s^i (\delta_{ps} - 1), \text{ in } M,
\]

where \( h_j^* > 0 \), \( K = (K_{ij}) \) is the Cartan matrix of the Lie algebra \( \mathfrak{g} \).

- If the rank of the simple Lie algebra is 2, \( \exists \) three types of corresponding Cartan matrices of rank 2:

\[
A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad B_2 = C_2 = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.
\]

In general, \( \exists \) four types of simple non-exceptional Lie Algebra: 
\( A_m, B_m, C_m, D_m \), whose Cartan subalgebra are \( sl(m+1), so(2m+1), sp(m), so(2m) \).

Corresponding to each of the four types of Lie Algebra, there is a Toda system.
Background

- Solutions of Toda systems are closely related to holomorphic curves in projective spaces.

From the classical Plücker formula, any holomorphic curve gives rise to a solution of $A_m$ type Toda system and the branch points of these curves corresponds to the singularities of the solutions.

On the other hand, if we integrate the $A_m$ Toda system, a solution defines a holomorphic curve in $\mathbb{CP}^n$ at least locally.

- In [S.S. Chern, J.G. Wolfson (1987)], [K. Uhlenbeck (1989)], it was noticed that the general Toda system is an integrable system. The integrability has been further discussed in [C.S. Lin, J. Wei, D. Ye (2012)].

- Algebraic geometry, modular forms, Painlevé VI equation, non-abelian Chern-Simons gauge field theory.
Our main goal

- We want to compute the Leray-Schauder topological degree of the following Toda system of rank two:

\[
\begin{cases}
\Delta u_1 + K_{11}\rho_1 \left( \frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1} d\nu_g} - 1 \right) + K_{12}\rho_2 \left( \frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2} d\nu_g} - 1 \right) = 0 \\
\Delta u_2 + K_{21}\rho_1 \left( \frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1} d\nu_g} - 1 \right) + K_{22}\rho_2 \left( \frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2} d\nu_g} - 1 \right) = 0
\end{cases}
\text{in } M \ (Toda),
\]

where \( h_i(x) = h_i^*(x) e^{-4\pi \sum_{p \in S_i} \alpha_{p,i} G(x,p)} \), \( h_i^* > 0 \), and

\[
K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad B_2 = C_2 = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad \text{or} \quad G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.
\]
A priori bounds of solutions for (Toda) in $\dot{H}^1(M)$

**Theorem (C.S. Lin, J. Wei, L. Zhang)**

Let $u = (u_1, u_2)$ be a solution of (Toda) with all $\alpha_{p,i} \in \mathbb{N}$.

Suppose $\rho_1, \rho_2 \notin 4\pi \mathbb{N}$. Then,

$$\|u_1\|_{L^\infty} + \|u_2\|_{L^\infty} \leq C = C(\rho_i, h_i, \alpha_{p,i}, M).$$
Leray-Schauder degree \( d_\rho \)

- By [C.S. Lin, J. Wei, L. Zhang], we can define the Leray-Schauder degree \( d^K_{\rho_1, \rho_2} \) of \((Toda)\) for \( \rho_1 \in (4i\pi, 4(i + 1)\pi) \) and \( \rho_2 \in (4j\pi, 4(j + 1)\pi) \), \( i, j \in \mathbb{N} \cup \{0\} \).

Since the degree \( d^K_{\rho_1, \rho_2} \) is a homotopic invariant for \( \rho_1 \in (4i\pi, 4(i + 1)\pi) \) and \( \rho_2 \in (4j\pi, 4(j + 1)\pi) \), \( i, j \in \mathbb{N} \cup \{0\} \),

\[
d^K_{\rho_1, \rho_2} \equiv d^K_{i, j}.
\]

- The generating function

\[
g_i^{(2)}(x, K) = \sum_{j=0}^{\infty} d^K_{i, j} x^j.
\]

- By [C.C. Chen, C.S. Lin (2015)],

\[
g^{(2)}_0(x) = (1 + x + \cdots)^{1-\chi(M)} \prod_{p \in S_2} (1 + x + \cdots + x^{\alpha_p}).
\]
Main result I: without singularity

• Let \( h_i(x) = h_i^*(x)e^{-4\pi \sum_{p \in S_i} \alpha_{p,i} G(x,p)} \) where \( h_i^* > 0 \), i.e. \( h_i(x) = 0 \iff x \in S_i \).

• \( d_{1,j}^K \): the Leray-Schauder degree of \((Toda)\) for \( \rho_1 \in (4\pi, 8\pi) \) and \( \rho_2 \in (4j\pi, 4(j + 1)\pi), j \in \mathbb{N} \cup \{0\} \).

Theorem (L., C. S. Lin, J. Wei, W. Yang)

Suppose \( S_1 \cup S_2 = \emptyset \) (i.e. \( h_1, h_2 > 0 \)). Then

\[
g_1^{(2)}(x, K) = \sum_{j=0}^{\infty} d_{1,j}^K x^j = (1 - x)^{\chi(M)-1} (1 - \chi(M)(1 + x + \cdots + x^{-K_{21}}))
\]
Main result II: with (simple) singularity

- Let $h_i(x) = h_i^*(x) e^{-4\pi \sum_{p \in S_i} \alpha_{p,i} G(x,p)}$ where $h_i^* > 0$,
i.e. $h_i(x) = 0 \iff x \in S_i$,

- $d_{1,j}^K$: the Leray-Schauder degree of $(Toda)$ for

$\rho_1 \in (4\pi, 8\pi)$ and $\rho_2 \in (4j\pi, 4(j+1)\pi), j \in \mathbb{N} \cup \{0\}$.

**Theorem (L., C. S. Lin, W. Yang, L. Zhang)**

*Suppose $\alpha_{p,i} \in \{0, 1, 2\}, \ i = 1, 2$. Then*

$$g_1^{(2)}(x, K) = \sum_{k=0}^{\infty} d_{1,k}^K x^k = (1 - x)^{\chi(M)-1} \left[ \prod_{p \in S_2} (1 + x + \cdots + x^{\alpha_{p,2}}) 
- (\chi(M) - |S_2 \cup S_1|) (1 + \cdots + x^{-K_{21}}) \prod_{p \in S_2} (1 + \cdots + x^{\alpha_{p,2}}) 
- \sum_{p \in S_2 \setminus S_1} (1 + x + \cdots + x^{\alpha_{p,2}-K_{21}}) \prod_{q \in S_2 \setminus \{p\}} (1 + x + \cdots + x^{\alpha_{q,2}}) \right].$$
Applications

- $\chi(M) = 2 - 2g$, where $g$ denotes the genus of $M$.

Corollary

Suppose $\alpha_{p,i} \in \{0, 1, 2\}, \ i = 1, 2$.

If $g > 0$, then (Toda) always has a solution when $\rho_1 \in (0, 4\pi) \cup (4\pi, 8\pi)$, $\rho_2 \notin 4\pi \mathbb{N}$.

$$
\begin{cases}
\Delta u_1^* + 2\rho_1 \left( \frac{e^{u_1^*}}{\int_M e^{u_1^*}} - 1 \right) - \rho_2 \left( \frac{e^{u_2^*}}{\int_M e^{u_2^*}} - 1 \right) = 4\pi \sum_{p \in S_1} \alpha_{p,1}(\delta_p - 1), \\
\Delta u_2^* + 2\rho_2 \left( \frac{e^{u_2^*}}{\int_M e^{u_2^*}} - 1 \right) - \rho_1 \left( \frac{e^{u_1^*}}{\int_M e^{u_1^*}} - 1 \right) = 4\pi \sum_{p \in S_2} \alpha_{p,2}(\delta_p - 1).
\end{cases}
$$

on $S^2$

Corollary

Suppose $S_1 = \emptyset$, $|S_2| = 1, 2$, and $\alpha_{p,2} = 1$ for any $p \in S_2$.

Then the above equation has a solution.
Applications

Suppose $h_1 > 0$ and $h_2 > 0$.
Let $K = A_2$, $\rho_1 \in (4\pi, 8\pi)$, and $\rho_2 \in (4k\pi, 4(k + 1)\pi)$.

Corollary

Let $M = \mathbb{S}^2$. Then

$$d_{1,k}^{A_2} = \begin{cases} 
-1, & \text{if } k = 0, \\
-1, & \text{if } k = 1, \\
2, & \text{if } k = 2, \\
0, & \text{if } k \geq 3.
\end{cases}$$

Remark

In [A. Jevnikar, S. Kallel, A. Malchiodi (2015)], it was proved that there exists a solution of (Toda) for any compact surface $M$.

So if $M = \mathbb{S}^2$ and $k \geq 3$, then (Toda) generically has more than one solution.
The idea of the proof

(i) to introduce the shadow system due to the bubbling phenomena when \( \rho_1 \) crosses \( 4\pi \) and \( \rho_2 \notin 4\pi \mathbb{N} \):

**Theorem**

Suppose \( \alpha_{p,i} \in \mathbb{N} \cup \{0\} \) and \((u_{1k}, u_{2k})\) are solutions of \((Toda)\) with \((\rho_{1k}, \rho_{2k}) \to (4\pi, \rho_2)\) satisfying \( \rho_2 \notin 4\pi \mathbb{N} \) and \( \max_M(u_{1k}, u_{2k}) \to +\infty \). Then

\[
\rho_{1k} \frac{h_1 e^{u_{1k}}}{\int_M h_1 e^{u_{1k}}} \to 4\pi \delta_Q, \quad Q \in M \setminus S_1, \quad \text{and}
\]

\[
u_{2k} \to w + 4\pi K_{21} G(x, Q) \quad \text{in } C^{2,\alpha}_{loc}(M \setminus \{Q\}),
\]

where \((w, Q)\) is a solution of

\[
\begin{cases}
\Delta w + 2\rho_2 \left( \frac{h_2 e^{w+4\pi K_{21} G(x, Q)}}{\int_M h_2 e^{w+4\pi K_{21} G(x, Q)}} - 1 \right) = 0, \\
\nabla \left( \log h_1 e^{\frac{K_{12}}{2} w} \right) \big|_{x=Q} = 0, \quad \text{and } Q \notin S_1,
\end{cases}
\]

(Shadow)
The idea of the proof

(ii) to show how to calculate the topological degree of Toda systems by computing the topological degree of the shadow systems:

- $d_j^S$: the Leray-Schauder degree for (Shadow) when $\rho_2 \in (4j\pi, 4(j + 1)\pi)$.

Theorem

$$d_{1,j}^K - d_{0,j}^K = -d_j^S.$$

(iii) to calculate the topological degree of the shadow system for one point blow up.
Thank you for your attention!