Quasi-neutral limit for Euler-Poisson system in the presence of plasma sheaths

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Euler-Poisson equations

The motion of the positive ions in a plasma is governed by

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + K \nabla \rho &= \rho \nabla \phi, \\
\varepsilon \Delta \phi &= \rho - e^{-\phi},
\end{align*}
\]

- \(\rho(x, t), u(x, t)\) and \(-\phi(x, t)\) represent the density, velocity of the positive ions and the electrostatic potential, respectively.
- \(K\) = temperature of positive ion
- \(\sqrt{\varepsilon}\) is the ratio of the Debye length \(\lambda_D\) to the radius of an inner sphere \(R\), that is, \(\varepsilon = \lambda_D^2/R^2\). (Lab setting \(\lambda_D \sim 10^{-4}\))
Plasma physics

- Plasma (in Greek “anything formed”) is one of the four fundamental states of matter.

- Like gas, plasma does not have a definite shape or a definite volume unless enclosed in a container.

- Unlike gas, under the influence of a magnetic field, it may form structures such as filaments, beams and double layers.

- The presence of a significant number of charge carriers makes plasma electrically conductive so that it responds strongly to electromagnetic fields.
Plasma physics

- **Quasi-neutrality** (quasi, from the Latin, “as if”, “resembling”) describes the apparent charge neutrality of a plasma overall, while at smaller scales, the positive and negative charges making up the plasma, may give rise to charged regions and electric fields.

- Since electrons are very mobile, plasmas are excellent conductors of electricity, and any charges that develop are readily neutralized, and in many cases, plasmas can be treated as being electrically neutral.

- The distance over which quasi-neutrality may break down, is often described by the Debye length (or Debye sphere), and varies according to the physical characteristics of the plasma. The Debye length is typically less than a millimetre (ie. charged regions will not exceed a millimetre), in plasmas found in fluorescent light tubes, tokamaks (used in fusion research), and the ionosphere.
From the two-fluid model to our model

Full two-fluid model reads as

\[ \partial_t n_{\pm} + \nabla \cdot (n_{\pm} v_{\pm}) = 0, \]
\[ n_{\pm} m_{\pm} (\partial_t v_{\pm} + (v_{\pm} \cdot \nabla)v_{\pm}) + T_{\pm} \nabla n_{\pm} = \pm en_{\pm} \nabla \phi, \]
\[ \varepsilon \Delta \phi = 4\pi e (n_+ - n_-) \]

After non-dimensionalization together with Boltzmann-Maxwell relation \((m_- \ll m_+)\), one obtains EP for ion dynamics:

\[ \rho_t + \nabla \cdot (\rho u) = 0, \]
\[ (\rho u)_t + \nabla \cdot (\rho u \otimes u) + K \nabla \rho = \rho \nabla \phi, \]
\[ \varepsilon \Delta \phi = \rho - e^{-\phi}. \]
Euler equations of compressible fluids:

- Euler equations of compressible fluids is given by
  \[ \rho_t + \nabla \cdot (\rho u) = 0, \]
  \[ \rho(u_t + (u \cdot \nabla)u) + \nabla p = 0, \]
  \[ S_t + (u \cdot \nabla)S = 0, \]

where \( P(\rho, S) = A\rho^\gamma e^S \) is the pressure and \( \gamma > 1 \) is the adiabatic index.

\[ \sigma := \sqrt{P_{\rho}(\bar{\rho}, \bar{S})} = \sqrt{A\gamma\bar{\rho}^{\gamma-1}e^\bar{S}} \] is the speed of sound.

- Note that the maximum speed of propagation of the wave of a smooth disturbance is governed by \( \sigma \).

- Heuristically, the linear approximation of the Euler equations near the constant state \((\rho, u, S) = (\bar{\rho}, 0, \bar{S})\) is the wave equation:
  \[ \rho_{tt} - \sigma^2 \Delta \rho = 0. \]

- (T. Sideris 1985) \( C^1 \) solutions may blow up in finite-time.
  - Hyperbolic + Nonlinear \( \Rightarrow \) blow up
Dispersion relation

EP for ion dynamics:
\[
\begin{align*}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + K \nabla \rho^\gamma &= \rho \nabla \phi, \\
\varepsilon \Delta \phi &= \rho - e^{-\phi}.
\end{align*}
\]

Linearizing about the constant state \((\rho, u, \phi) = (1, 0, 0)\), we have
\[
\begin{align*}
\rho_t + \nabla u &= 0, \\
u_t + K \nabla \rho &= \nabla \phi, \\
\varepsilon \Delta \phi &= \rho + \phi.
\end{align*}
\]

Furthermore we obtain the wave equation with nonlocal term:
\[
\rho_{tt} - K \Delta \rho = -\Delta((1 + \Delta)^{-1} \rho).
\]

Dispersion relation:
\[
\lambda = i |\xi| \sqrt{\frac{K + 1 + K |\xi|^2}{1 + |\xi|^2}}
\]
Plasma sheaths

- Physical situation: a negatively charged material is immersed in a plasma and the plasma contacts with its surface.
- The plasma sheath is the transition from a plasma to a solid surface, where the positive ions outnumber the electrons.
- (Langmuir, 1929) Plasma sheath is emerged.
- The interaction of electron and positive ion space charges in cathode sheaths, Phys. Rev. 33 (1929), pp. 954–989.
- (Bohm, 1949) Bohm criterion.
Plasma sheaths

- The plasma sheath is a layer in a plasma which has a greater density of positive ions, and hence an overall excess positive charge, that balances an opposite negative charge on the surface of a material with which it is in contact.
Program: hydrodynamic description of the formation of plasma sheaths via the Euler-Poisson system

- Plasma sheaths (sharp transition layers near the boundary, stationary boundary layer solutions)
- Bohm criterion (physical criterion)
- Existence of the stationary solution (plasma sheath)
  - Euler-Poisson system
  - its quasi-neutral limiting equations (Euler system)
- Stability (time-asymptotic) vs blow-up
  - Local existence (hyperbolic initial-boundary value problem)
  - Global existence (uniform estimates) or Spectral analysis towards blow-up
- Quasi-neutral limit
  - Stationary and time-dependent problem
  - Uniform decay estimate (in terms of $\varepsilon = \text{Debye length}$)
  - Correctors (information on boundary layers)
Euler-Poisson system and its quasi-neutral limiting eqns

E-P system:

\[
\begin{align*}
\rho^\varepsilon_t + \nabla \cdot (\rho^\varepsilon u^\varepsilon) &= 0, \\
(\rho^\varepsilon u^\varepsilon)_t + \nabla \cdot (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + K\nabla \rho^\varepsilon &= \rho^\varepsilon \nabla \phi^\varepsilon, \\
\varepsilon \Delta \phi^\varepsilon &= \rho^\varepsilon - e^{-\phi^\varepsilon}.
\end{align*}
\]

The associated quasi-neutral limiting equations (\(\varepsilon = 0\))

\[
\begin{align*}
\rho^0_t + \nabla \cdot (\rho^0 u^0) &= 0, \\
u^0_t + u^0 \cdot \nabla u^0 + (K + 1) \nabla \log \rho^0 &= 0.
\end{align*}
\]
Euler-Poisson system in an annular domain

- Annulus domain $\Omega$:

$$\Omega := \{(x_1, x_2, x_3) \in \mathbb{R}^3 | 1 < \sum_{j=1}^3 x_i^2 < 1 + L\},$$

where $L > 0$ is the width of the annulus domain.

We consider E-P in $\Omega$:

$$\rho_t + \nabla \cdot (\rho u) = 0,$$
$$\rho u_t + \nabla \cdot (\rho u \otimes u) + K \nabla \rho = \rho \nabla \phi,$$
$$\varepsilon \Delta \phi = \rho - e^{-\phi},$$

where $x \in \Omega$ and $t \geq 0$.

- Initial conditions:

$$(\rho, u)(0, x) = (\rho_0, u_0)(x), \quad \inf_{x \in \Omega} \rho_0(x) > 0,$$
$$\inf_{x \in \Omega} (|u_0|^2 - K \rho_0)(x) > 0, \quad \sup_{x \in \Omega} u_0 \cdot \nu(x) < 0,$$

- Boundary conditions:

$$(\rho, u)(t, 1 + L, \omega) = (1, u_+ \omega), \quad \omega := \frac{x}{|x|},$$
$$\phi(t, 1, \omega) = \phi_b, \quad \phi(t, 1 + L, \omega) = 0.$$

where $1$, $u_+$ and $\phi_b$ are constants.
Existence of stationary radial solutions

Condition (Bohm’s Criterion for the annulus domain)

\[ u^2_+ > a^* (1 + K), \]

where \( a^* > 1 \) is a larger root of \( 1 + 4 \log(1 + L) - x + \log x = 0 \).

Theorem (Existence for the stationary Q-N-L)

Let \( u^2_+ > a^* (1 + K) \) hold. Then the SLP has a unique solution

\[ (\tilde{\rho}^0, \tilde{u}^0) \in C^\infty (I) \cap C(\bar{I}). \]

Moreover, \( \tilde{\rho}^0 \) is a monotone decreasing function with the bound

\[ 1 = \tilde{\rho}^0(1 + L) \leq \tilde{\rho}^0(r) \leq \tilde{\rho}^0(1). \]

Theorem (Existence for the stationary E-P eqns.)

There are constants \( \varepsilon_2 > 0 \) and \( \delta_1 > 0 \) such that if \( \varepsilon \leq \varepsilon_2 \) and \( \phi_b \leq - \log \tilde{\rho}^0(1) + \delta_1 \), the SEP has a unique solution \( \tilde{\phi} \in C^2 ([1, 1 + L]) \) satisfying

- \( \tilde{\phi}(r) \geq G(r, \tilde{\rho}^0_*(r)) \quad \forall r \in [1, 1 + L] \)
- \( \tilde{\phi}(r) \leq - \log \tilde{\rho}^0(r) + \delta_1 \quad \forall r \in [1, 1 + L] \)
Quasi-neutral limit in the presence of plasma sheaths

Theorem (Quasi-neutral limit)
Under the same assumptions, the difference $(\tilde{\rho} - \tilde{\rho}^0 - \theta_\phi^0, \tilde{u} - \tilde{u}^0 - \theta_u^0, \tilde{\phi} - \tilde{\phi}^0 - \theta_\phi^0)$ satisfies the decay estimates
\[
\|(\tilde{\rho} - \tilde{\rho}^0 - \theta_\rho^0, \tilde{u} - \tilde{u}^0 - \theta_u^0, \tilde{\phi} - \tilde{\phi}^0 - \theta_\phi^0)\|_{L^2} \leq C\varepsilon^{3/4},
\]
\[
\|(\tilde{\rho} - \tilde{\rho}^0 - \theta_\rho^0, \tilde{u} - \tilde{u}^0 - \theta_u^0, \tilde{\phi} - \tilde{\phi}^0 - \theta_\phi^0)\|_{H^1} \leq C\varepsilon^{1/4},
\]
\[
\|(\tilde{\rho} - \tilde{\rho}^0 - \theta_\rho^0, \tilde{u} - \tilde{u}^0 - \theta_u^0, \tilde{\phi} - \tilde{\phi}^0 - \theta_\phi^0)\|_{L^\infty} \leq C\varepsilon^{1/2},
\]
where $C > 0$ is a constant independent of $\varepsilon$.

Proposition (pointwise estimates for the correctors)
Under the same assumptions, for any $r \in \bar{T}$, there hold
\[
|\phi_b - \tilde{\phi}^0(1)|\left(e^{-\sqrt{c_0/\varepsilon}(r-1)} - e^{-\sqrt{c_0/\varepsilon}L}\right) \leq |\theta_\phi^0(r)| \leq |\phi_b - \tilde{\phi}^0(1)|e^{-\sqrt{c_0/\varepsilon}(r-1)},
\]
where $c$, $C$, $c_0$ and $C_0$ are some positive constants independent of $\varepsilon$.

- This implies that the thickness of the boundary layer is of order $O(\sqrt{\varepsilon}) = O(\lambda_D)$.
- By a direct calculation, it seems that the best $L^2$ and $H^1$ estimates for $\tilde{\phi} - \tilde{\phi}^0$ are
\[
\|\tilde{\phi} - \tilde{\phi}^0\|_{L^2} \leq C\varepsilon^{1/4} \quad \text{and} \quad \|\tilde{\phi} - \tilde{\phi}^0\|_{H^1} \leq C\varepsilon^{-1/4},
\]
respectively.
Numerical experiments in radial symmetry setting (Jung-K-Suzuki 2016)

Figure 2: Graphs of solutions, (a) $\tilde{\phi}$, $\tilde{\phi}^0$, $\theta_{\phi}^0$, (b) $\tilde{\rho}$, $\tilde{\rho}^0$, $\theta_{\rho}^0$, (c) $\tilde{u}$, $\tilde{u}^0$, $\theta_{u}^0$. 
Existence for the stationary Euler-Poisson system

**Condition**
Recall the “physical” assumptions:
- (Bohm criterion) \( u^2 > a^* (1 + K) \)
- \( \phi_b \geq G(1, \tilde{\rho}^0(1)) \)

**Theorem**

There are constants \( \varepsilon_2 > 0 \) and \( \delta_1 > 0 \) such that if \( \varepsilon \leq \varepsilon_2 \) and \( \phi_b \leq -\log \tilde{\rho}^0(1) + \delta_1 \), the SEP has a unique solution \( \tilde{\phi} \in C^2([1, 1 + L]) \) satisfying

- \( \tilde{\phi}(r) \geq G(r, \tilde{\rho}^0(r)) \quad \forall r \in [1, 1 + L] \)
- \( \tilde{\phi}(r) \leq -\log \tilde{\rho}^0(r) + \delta_1 \quad \forall r \in [1, 1 + L] \)
Quasi-neutral limit problem

Study the quasi-neutral limit problem ($\varepsilon \to 0$):

For ($\rho = \rho^\varepsilon$, $u = u^\varepsilon$, $\phi = \phi^\varepsilon$),

$$
\begin{aligned}
(\rho u)_r &= -\frac{2}{r}\rho u, \\
(\rho u^2 + K\rho)_r &= \rho \phi_r - \frac{2}{r}\rho u^2, \\
\varepsilon r^{-2}(r^2 \phi_r)_r &= (\rho - e^{-\phi}), \\
\phi &= \phi_b \text{ at } r = 1, \\
(\rho, u, \phi) &= (1, u_+, 0) \text{ at } r = 1 + L.
\end{aligned}
$$

(0.1)

For ($\rho = \rho^0$, $u = u^0$, $\phi = \phi^0$),

$$
\begin{aligned}
(\rho u)_r &= -\frac{2}{r}\rho u, \\
(\rho u^2 + K\rho)_r &= \rho \phi_r - \frac{2}{r}\rho u^2, \\
0 &= \rho - e^{-\phi}, \\
(\rho, u) &= (1, u_+) \text{ at } r = 1 + L.
\end{aligned}
$$

(0.2)

- discrepancy bet’n the boundary data $\phi_b$ and $-\log \bar{\rho}(1)$ causes a sharp transition layer
Quasi-neutral limit in the presence of plasma sheaths

Theorem (Quasi-neutral limit)

Under the same assumptions, the difference \((\tilde{\rho} - \tilde{\rho}^0 - \theta_\phi^0, \tilde{u} - \tilde{u}^0 - \theta^0_u, \tilde{\phi} - \tilde{\phi}^0 - \theta^0_\phi)\) satisfies the decay estimates

\[
\| (\tilde{\rho} - \tilde{\rho}^0 - \theta_\rho^0, \tilde{u} - \tilde{u}^0 - \theta^0_u, \tilde{\phi} - \tilde{\phi}^0 - \theta^0_\phi) \|_{L^2} \leq C\varepsilon^{3/4},
\]

\[
\| (\tilde{\rho} - \tilde{\rho}^0 - \theta_\rho^0, \tilde{u} - \tilde{u}^0 - \theta^0_u, \tilde{\phi} - \tilde{\phi}^0 - \theta^0_\phi) \|_{H^1} \leq C\varepsilon^{1/4},
\]

\[
\| (\tilde{\rho} - \tilde{\rho}^0 - \theta_\rho^0, \tilde{u} - \tilde{u}^0 - \theta^0_u, \tilde{\phi} - \tilde{\phi}^0 - \theta^0_\phi) \|_{L^\infty} \leq C\varepsilon^{1/2},
\]

where \(C > 0\) is a constant independent of \(\varepsilon\), and \(\theta^0\) is a corrector which is a solution to the dominating equations:

\[
\varepsilon \theta^0_{rr} = g(r, \phi^0 + \theta^0) - e^{-\phi^0 - \theta^0},
\]

\[
\theta^0(1) = \phi_b + \log \rho^0(1), \quad \theta^0(1 + L) = 0.
\]

RMK. By a direct calculation, it seems that the best \(L^2\) and \(H^1\) estimates for \(\tilde{\phi} - \tilde{\phi}^0\) are

\(\| \tilde{\phi} - \tilde{\phi}^0 \|_{L^2} \leq C\varepsilon^{1/4}\) and \(\| \tilde{\phi} - \tilde{\phi}^0 \|_{H^1} \leq C\varepsilon^{-1/4}\), respectively.
Proposition (pointwise estimates)

Under the same assumptions, for any $r \in \bar{I}$, there hold

$$|\phi_b - \tilde{\phi}^0(1)| \left(e^{-\sqrt{C_0/\varepsilon}(r-1)} - e^{-\sqrt{c_0/\varepsilon L}}\right) \leq |\theta^0_\phi(r)| \leq |\phi_b - \tilde{\phi}^0(1)| e^{-\sqrt{c_0/\varepsilon}(r-1)},$$

where $c, C, c_0$ and $C_0$ are some positive constants independent of $\varepsilon$.

- This implies that the thickness of the boundary layer is of order $O(\sqrt{\varepsilon}) = O(\lambda_D)$. 
Numerical Experiments for Quasi-neutral limit problem

Figure 2: Graphs of solutions, (a) $\tilde{\phi}, \tilde{\phi}^0, \theta^0_{\phi}$, (b) $\tilde{\rho}, \tilde{\rho}^0, \theta^0_{\rho}$, (c) $\tilde{u}, \tilde{u}^0, \theta^0_u$. 
Sketch of proof

To find the boundary layer structures, we decompose \( \tilde{\phi} = \tilde{\phi}^0 + \theta \).

\[
\varepsilon \frac{d^2}{dr^2} (\tilde{\phi}^0 + \theta) + \frac{2}{r} \varepsilon \frac{d}{dr} (\tilde{\phi}^0 + \theta) = g(r, \tilde{\phi}^0 + \theta) - e^{-\tilde{\phi}^0 - \theta}.
\]

Since \( \tilde{\phi}^0 \) is independent of \( \varepsilon \), the terms \( \varepsilon d^2 \tilde{\phi}^0 / dr^2 \) and \( \varepsilon d\tilde{\phi}^0 / dr \) are small and we drop them. In addition, to zoom the behavior of \( \theta \) near \( r = 1 \), using a stretched variable \( \bar{r} = (r - 1) / \sqrt{\varepsilon} \), we obtain

\[
\frac{d^2}{d\bar{r}^2} \theta + \frac{2}{\sqrt{\varepsilon} \bar{r} + 1} \sqrt{\varepsilon} \frac{d}{d\bar{r}} \theta = g(r, \tilde{\phi}^0 + \theta) - e^{-\tilde{\phi}^0 - \theta}.
\]  (0.3)

Then we look for the leading asymptotic term \( \theta_0^0 \) with \( \theta \sim \theta_0^0 + \sqrt{\varepsilon} \theta_1^0 \). Substituting this expansion for \( \theta \) in (0.3) and using the Taylor expansion for \( g(r, \phi) \) and \( e^\phi \) about \( \phi = \tilde{\phi}^0 + \theta_0^0 \), we find that

\[
\frac{d^2}{d\bar{r}^2} \theta_0^0 = g(r, \tilde{\phi}^0 + \theta_0^0) - e^{-\tilde{\phi}^0 - \theta_0^0} + \mathcal{O}(\sqrt{\varepsilon}).
\]

At the leading order, i.e. \( \mathcal{O}(1) \), we obtain the equation for the boundary layers.
Sketch of proof

- Define a corrector function by
  \[
  \tilde{\theta}(r) := (\phi_b - \tilde{\phi}^0(1)) \exp \left( - \frac{r - 1}{\sqrt{\epsilon}} \right) \chi(r).
  \]

- Then \( \theta_0^0 - \tilde{\theta} \) satisfies
  \[
  \epsilon (\theta_0^0 - \tilde{\theta})'' = g(r, \tilde{\phi}^0 + \theta_0^0) - e^{-\tilde{\phi}^0 - \theta_0^0} - \tilde{\theta} - R_1, \tag{0.4a}
  \]
  \[
  (\theta_0^0 - \tilde{\theta})(1) = (\theta_0^0 - \tilde{\theta})(1 + L) = 0, \tag{0.4b}
  \]
  where \( R_1 := (-2\sqrt{\epsilon} \chi' + \epsilon \chi'') (\phi_b - \tilde{\phi}^0(1)) e^{-(r-1)/\sqrt{\epsilon}}. \)

- On the other hand, by a straightforward calculation, one can estimate \( \tilde{\theta} \) and \( R_1 \) as
  \[
  \|\tilde{\theta}\|_{L^2} + \|R_1\|_{L^2} \leq C \epsilon^{1/4}, \tag{0.5}
  \]
  where \( C > 0 \) is a constant independent of \( \epsilon \).
• By $L^2$ estimate in a straightforward fashion, we have
\[
\varepsilon \int_1^{1+L} |(\theta_0^\phi - \tilde{\theta})'|^2 dr + \int_1^{1+L} \theta_0^\phi \left( g(r, \tilde{\phi}^0 + \theta_0^\phi) - e^{-\tilde{\phi}^0 - \theta_0^\phi} \right) dr \\
= \int_1^{1+L} \tilde{\theta} \left( g(r, \tilde{\phi}^0 + \theta_0^\phi) - e^{-\tilde{\phi}^0 - \theta_0^\phi} - g(r, \tilde{\phi}^0) + e^{-\tilde{\phi}^0} \right) dr + \int_1^{1+L} (\tilde{\theta} + R_1)(\theta_0^\phi - \tilde{\theta}) dr.
\]

• This gives
\[
\sqrt{\varepsilon} \| \theta_0^\phi - \tilde{\theta} \|^2_{L^2} + \| \theta_0^\phi \|^2_{L^2} \leq C\varepsilon^{1/4}.
\]

• It is easy to check that
\[
\| \tilde{\theta}' \|^2_{L^2} \leq C\varepsilon^{-1/4}
\]

• Combining these, we have
\[
\sqrt{\varepsilon} \| \theta_0^\phi \|^2_{L^2} + \| \theta_0^\phi \|^2_{L^2} \leq C\varepsilon^{1/4}.
\]
Moreover, define
\[ w := \tilde{\phi} - \tilde{\phi}^0 - \theta^0. \]
Then it satisfies
\[ \varepsilon (r^2 w')' = r^2 \{ K(r, \tilde{\phi}, \phi^0 + \theta^0) + J(r, \tilde{\phi}, \phi^0 + \theta^0) \} w + R_2, \]
\[ w(1) = w(1 + L) = 0, \]
where \( R_2 = -\varepsilon \{ 2r(\tilde{\phi}^0 + \theta^0)' + r^2 (\tilde{\phi}^0)'' \} \).

Note that
\[ \| R_2 \|_{L^2} \leq C \varepsilon^{3/4}. \]

Again, by energy estimate, we arrive at
\[ \varepsilon \| w' \|_{L^2}^2 + \| w \|_{L^2}^2 \leq C \| R_2 \|_{L^2}^2 \leq C \varepsilon^{3/2}. \]
Thank you for your attention